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LETTER TO THE EDITOR

Modulated Taylor–Couette flow as a dynamical system

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Abstract. A Lorenz-like truncation of the full hydrodynamic equations is used to study the onset of first instability of modulated Taylor-Couette flow. The linear stability analysis is carried out on the lowest order (2×2) non-autonomous dynamical system. The use of good wavefunctions for the truncation yields qualitative agreement with exact hydrodynamics even at this low order of stability analysis.

The effect of modulation of control parameters on the onset of hydrodynamic instability has attracted considerable attention lately because of the sophistication of experimental techniques. The onset of thermally driven instability in the Rayleigh-Bénard geometry has been extensively studied. The full hydrodynamic equations (Venezian 1969, Rosenblat and Tanaka 1971), a low-order truncation thereof (Ahlers *et al* 1985) and amplitude equations (Ahlers *et al* 1985) have been studied and the results are in excellent agreement with each other. The full hydrodynamic equations for the modulated Taylor-Couette flow have been studied analytically by Hall (1975) and numerically by Riley and Laurence (1976). While Hall (1975) finds destabilisation of flow due to modulation for all frequencies, Riley and Laurence (1976) notice that, for high amplitudes of modulation, the effect of modulation is to destabilise the flow for low frequencies and stabilise it for high frequencies. For low amplitudes, Riley and Laurence (1976) do not notice the stabilisation at high frequencies perhaps because the effect at high frequencies is very small (it is barely within the range of numerical accuracy for moderate amplitudes and beyond it for smaller amplitudes). The experiments of Donnelly (1964) show stabilisation at all frequencies, while the more accurate experiment of Thompson (1968) shows a destabilisation at low frequencies and a stabilisation in the higher range.

Lately there has been some effort to introduce a truncated model for the Taylor-Couette problem similar to the Lorenz model for Rayleigh-Bénard geometry to understand the modulation effects. Kuhlmann (1985) used a truncated model for the modulated Taylor-Couette system using idealised boundary conditions and found that the simplest truncation yields stabilisation at all frequencies. To find destabilisation one has to include more modes. A truncation of the unmodulated system using the proper rigid boundary conditions has been carried out by Hshieh and Chen (1984). We carry out a similar truncation for the modulated system and find that *the lowest order* (two modes for the linearised system) *truncation leads to destabilisation* at low frequencies. At high frequencies the model shows stabilisation. This is contrary to Hall and we believe the result is not an artefact of the truncation.

The hydrodynamic equations for the linear stability analysis of modulated Taylor-Couette flow in the narrow gap approximation are

$$(D^2-a^2)(D^2-a^2-\partial/\partial\tau)u=v\left(1-z+\varepsilon\,\mathrm{Re}\,\frac{\sinh\alpha d(1-z)}{\sinh\alpha d}\,\mathrm{e}^{\mathrm{i}\omega t}\right)\tag{1a}$$

$$(D^2-a^2-\partial/\partial\tau)v=-Ta^2\left(1+\varepsilon\,\mathrm{Re}\,\frac{\alpha d}{\sinh\alpha d}\cosh\alpha d(1-z)\,\mathrm{e}^{\mathrm{i}\omega t}\right)u\tag{1b}$$

where u is the radial velocity, v the angular velocity, $z=r/d$, where r is the radial coordinate and d the gap, a the dimensionless wavenumber in the z direction, T the Taylor number, $\alpha=(\mathrm{i}\omega/\nu)^{1/2}$, τ the time in units of d^2/ν and $D=\mathrm{d}/\mathrm{d}z$. Axisymmetric flow is assumed. The boundary conditions are $v=0=u=Du$ at $z=0$ and $z=1$. For $\varepsilon=0$ (the unmodulated system), it has been shown by Chandrasekhar (1961) that the solutions

$$v_0=\sin\pi z\tag{2a}$$

and

$$u_0=A\sinh az+B\cosh az+Cz\sinh az+Dz\cosh az$$
$$+\frac{(1-z)\sin\pi z}{(\pi^2+a^2)^2}-\frac{4\pi}{(\pi^2+a^2)^3}\cos\pi z\tag{2b}$$

(the constants A, B, C and D can be found in Chandrasekhar) yield the critical Taylor number T_c to within one per cent of the exact numerical answer ($T_c=3430$, $a=3.11$). We use the functions $u_0(z)$ and $v_0(z)$ for the two-mode truncation of the hydrodynamic equations and set

$$u(z,t)=x(t)u_0(z)\tag{3a}$$

$$v(z,t)=y(t)v_0(z).\tag{3b}$$

Inserting the above forms in equations (1a) and (1b), and integrating over z after multiplying by $v_0(z)$ on either side, we obtain (Hshieh and Chen 1984, Kuhlmann 1985)

$$\dot{x}=-A_{11}x+A_{12}y\tag{4a}$$

$$\dot{y}=-B_{11}y+B_{12}x\tag{4b}$$

where

$$A_{11}=T_ca^2/2(\pi^2+a^2)^2\tag{5a}$$

$$A_{12}=A_{11}+T_ca^2(\pi^2+a^2)^{-2}\varepsilon\,\mathrm{Re}\,G(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\tag{5b}$$

$$B_{11}=\pi^2+a^2\tag{5c}$$

and

$$B_{12}=(\pi^2+a^2)(T/T_c)+(\pi^2+a^2)(T/2T_c)\varepsilon\,\mathrm{Re}\,F(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\left(\int_0^1 u_0(z)v_0(z)\,\mathrm{d}z\right)^{-1}.\tag{5d}$$

Note that

$$T_c=[(\pi^2+a^2)/a^2]\left(\int_0^1 u_0(z)v_0(z)\,\mathrm{d}z\right)^{-1}\tag{6}$$

a is the critical wavenumber which is equal to 3.11,

$$G(\omega)=2\int_0^1 v_0(z)\frac{\sinh \alpha d(1-z)}{\sinh \alpha d}\sin \pi z\,\mathrm{d}z \tag{7}$$

and

$$F(\omega)=2\int_0^1 u_0(z)\frac{\alpha d}{\sinh \alpha d}\cosh \alpha d(1-z)\sin \pi z\,\mathrm{d}z. \tag{8}$$

Defining $r=T/T_c$, rescaling time by the factor π^2+a^2 and introducing

$$\tilde{\varepsilon}=\varepsilon F(\omega)\left(2\int_0^1 u_0(z)v_0(z)\,\mathrm{d}z\right)^{-1}=\varepsilon\gamma \tag{9}$$

$$\alpha=T_c a^2/2(\pi^2+a^2)^3 \tag{10}$$

we arrive at the system of equations

$$\dot{x}=-\alpha(x-y)+\mathrm{Re}(\tilde{\varepsilon}\beta\,\mathrm{e}^{\mathrm{i}\omega t})y \tag{11a}$$

$$\dot{y}=-y+rx+\mathrm{Re}(\tilde{\varepsilon}\,\mathrm{e}^{\mathrm{i}\omega t})rx \tag{11b}$$

where

$$\beta=2\alpha\frac{\pi^2+a^2}{T_c a^2}\frac{G(\omega)}{F(\omega)}. \tag{12}$$

We note that evaluation of $G(\omega)$ and $F(\omega)$ in the limiting cases of $\omega\simeq 0$ and $\omega\gg 1$ leads to

$$\beta(\omega\to 0)\to\alpha \tag{13a}$$

$$\beta(\omega\gg 1)\to 0.03\alpha. \tag{13b}$$

For $\varepsilon=0$ (no modulation), equations (11a) and (11b) show that the onset of instability occurs at $r=r_0=1$. The problem is to determine the shift in the critical value of r in the presence of modulation ($\varepsilon\neq 0$).

We proceed perturbatively by expanding

$$x=x_0+\varepsilon x_1+\varepsilon^2 x_2+\ldots \tag{14a}$$

$$y=y_0+\varepsilon y_1+\varepsilon^2 y_2+\ldots \tag{14b}$$

$$r_c=r_0+\varepsilon r_1+\varepsilon^2 r_2+\ldots. \tag{14c}$$

Inserting equations (14a)-(14c) into equations (11a) and (11b) and equating terms of the same order in ε on either side, we obtain

$$L\begin{pmatrix}x_0\\ y_0\end{pmatrix}=0 \tag{15a}$$

$$L\begin{pmatrix}x_1\\ y_1\end{pmatrix}=\mathrm{Re}\begin{pmatrix}\gamma\beta y_0\,\mathrm{e}^{\mathrm{i}\omega t}\\ r_1x_0+r_0x_0\gamma\,\mathrm{e}^{\mathrm{i}\omega t}\end{pmatrix} \tag{15b}$$

$$L\begin{pmatrix}x_2\\ y_2\end{pmatrix}=\mathrm{Re}\begin{pmatrix}\gamma\beta y_1\,\mathrm{e}^{\mathrm{i}\omega t}\\ r_2x_0+r_1x_1+r_0x_1\gamma\,\mathrm{e}^{\mathrm{i}\omega t}+r_1x_0\gamma\,\mathrm{e}^{\mathrm{i}\omega t}\end{pmatrix} \tag{15c}$$

with the matrix operator L given by

$$L = \begin{pmatrix} (\partial/\partial\tau)+\alpha & -\alpha \\ -r_0 & (\partial/\partial\tau)+1 \end{pmatrix}. \tag{16}$$

Noting that the unmodulated instability is stationary (i.e. x_0 and y_0 are time independent) and that the solvability condition (the inhomogeneous term must be orthogonal to the solution of the homogeneous equation) needs to be applied to equations (15*b*) and (15*c*), we arrive at $r_1 = 0$ and

$$r_2 = \frac{|\gamma|^2}{2} \operatorname{Re} \frac{(\alpha+\beta)^2 - \mathrm{i}\omega\beta(1+\alpha)}{\omega^2 + \mathrm{i}\omega(1+\alpha)} \tag{17}$$

leading to

$$r_2 = |\gamma|^2 \frac{(\alpha+\beta)^2 - \beta(\alpha+1)^2}{2\alpha[\omega^2 + (\alpha+1)^2]} \tag{18}$$

for real β.

We note that if $\beta = 0$ and $\alpha = \sigma$, we recover the result for the Rayleigh-Bénard convection which is exactly as it should be. The correction for $\omega \to 0$, after making use of equations (9) and (13*a*), is given by

$$r_2 = -\frac{(\alpha-1)^2}{2(\alpha+1)^2} = -0.07 \tag{19}$$

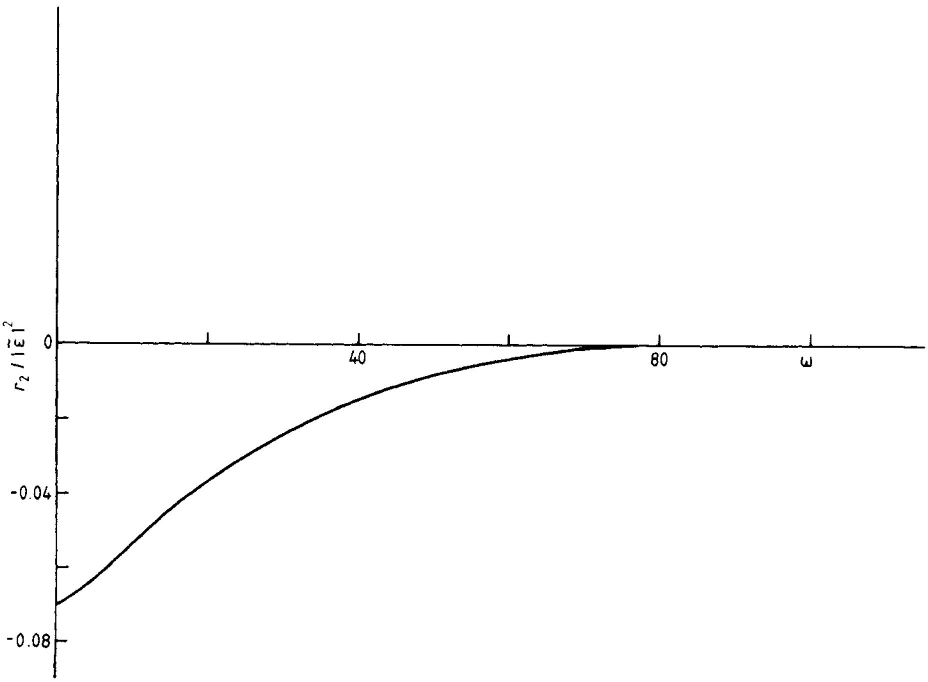

Figure 1. The first non-vanishing correction to the critical Taylor number due to modulation, normalised to its unmodulated value, as a function of frequency ω in units of ν/d^2. Note that for $\omega > 80$, the correction is positive (equation (18)) but the magnitude, being very small, is not visible on the scale of the drawing.

in excellent agreement with Hall. For very high frequencies the effect disappears asymptotically as

$$r_2 \simeq |\gamma|^2 \frac{0.9}{\omega^2} \tag{20}$$

as it does in exact hydrodynamics.

The sign of this asymptotic term, however, is in disagreement with Hall. To get the full course of the function, we evaluate the real and imaginary parts of β numerically from equation (12) and then make use of equation (17). This yields the curve shown in figure 1. We note that the crossing takes place at rather high frequency. The magnitude of the effect being extremely small in this range, numerical calculations are likely to fail. The disagreement with Hall (1975) is more serious. Could it be that the approximations $u_0(z)$ and $v_0(z)$ are too drastic in the high frequency regime? While more accurate wavefunctions are certainly desirable, we note that for very high frequencies the modulation effect in equation (1*a*) is small compared to that in equation (1*b*). This makes the situation essentially similar to the modulated Rayleigh-Bénard problem where stabilisation is known to occur at all frequencies.

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